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2004 J. Phys. A: Math. Gen. 37 347

(http://iopscience.iop.org/0305-4470/37/2/006)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 347–358 PII: S0305-4470(04)64995-X

# The $sl_2$ loop algebra symmetry of the twisted transfer matrix of the six-vertex model at roots of unity

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Received 19 June 2003
Published 15 December 2003
Online at stacks.iop.org/JPhysA/37/347 (DOI: 10.1088/0305-4470/37/2/006)

**Abstract**

We discuss a family of operators which commute or anti-commute with the twisted transfer matrix of the six-vertex model at $q$ being roots of unity, $q^{2N} = 1$. The operators commute with the Hamiltonian of the XXZ spin chain under the twisted boundary conditions, and they are also valid for the inhomogeneous case. For the case of the anti-periodic boundary conditions, we show explicitly that the operators generate the $sl_2$ loop algebra in the sector of the total spin operator $S^Z \equiv N/2 \,(\mathrm{mod}\ N)$. The infinite-dimensional symmetry leads to exponentially-large spectral degeneracies, as shown for the periodic boundary conditions (Deguchi T, Fabricius K and McCoy B M 2001 *J. Stat. Phys.* **102** 701). Furthermore, we derive explicitly the $sl_2$ loop algebra symmetry for the periodic XXZ spin chain with an odd number of sites in the sector $S^Z \equiv N/2 \,(\mathrm{mod}\ N)$ when $q$ is a primitive $N$th root of unity with $N$ odd. Interestingly, in the case of $N = 3$, various conjectures of combinatorial formulae for the XXZ spin chain with odd sites have been given by Stroganov and other authors. We also note a connection to the spectral degeneracies of the eight-vertex model.

PACS numbers: 75.10.Jm, 75.10.Pq, 05.50.+q

## 1. Introduction

The finite-size spectrum of the XXZ spin chain [1, 2] under twisted boundary conditions has attracted much interest recently and has been studied numerically or analytically such as by the Bethe ansatz [3–8]. The XXZ Hamiltonian defined on a ring of $L$ sites is given by

$$\mathcal{H}_{\mathrm{XXZ}} = J \sum_{j=1}^{L} \left( \sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y + \Delta \sigma_j^Z \sigma_{j+1}^Z \right). \tag{1}$$

Here $\sigma_j^\alpha$ for $\alpha = X, Y, Z$ denotes the Pauli matrix defined on the $j$th site. Under the periodic boundary conditions we have $\sigma_{L+1}^\alpha = \sigma_1^\alpha$ for any $\alpha$. We now introduce the twisted boundary conditions by

$$\sigma_{L+1}^{\pm} = \exp(\pm \mathrm{i}\Phi)\sigma_1^{\pm} \qquad \sigma_{L+1}^{Z} = \sigma_1^{Z}. \tag{2}$$

We call the parameter $\Phi$ the twist angle.

There are several physical applications of twisted boundary conditions. The twist angle corresponds to the magnetic flux threaded through the ring [9]. Taking the variation of eigenvalues under changes of the flux [10], the effective mass has been evaluated exactly for the many-body system of interacting fermions or bosons in one dimension which is equivalent to the XXZ spin chain [4]. The finite-size spectrum of the XXZ spin chain under the twisted boundary conditions has also been studied from the viewpoint of the finite-size analysis of conformal field theories [3]. Associated with the variation of the ground-state energy, the flows of excited states have been numerically investigated with respect to the twist angle [3, 4, 6, 7]. In the spectral flows, we find several level crossings with large degeneracy. However, it has not been discussed explicitly what kind of symmetry operators correspond to the spectral degeneracies. The question should be interesting in particular from the viewpoint of the violation of the level non-crossing rule as discussed by Heilmann and Lieb [11].

Recently, it has been found that the symmetry of the XXZ spin chain becomes enhanced at roots of unity. The $sl_2$ loop algebra commutes with the XXZ Hamiltonian under the periodic boundary conditions, and the infinite-dimensional algebra leads to many spectral degeneracies whose dimensions increase exponentially with respect to the system size [12]. Let us introduce the parameter $q$ through the XXZ anisotropy $\Delta$ as

$$\Delta = \frac{q + q^{-1}}{2}. \tag{3}$$

It is shown that the generators of the $sl_2$ loop algebra commute or anti-commute with the transfer matrix of the six-vertex model when $q^{2N} = 1$, and all the defining relations of the $sl_2$ loop algebra are explicitly derived for the case in the sector $S^Z \equiv 0 \,(\mathrm{mod}\ N)$. Here $S^Z$ denotes the $Z$ component of the total spin operator. Several aspects of the $sl_2$ loop algebra symmetry of the XXZ spin chain have been studied [13–19]. In particular, its connection to the spectral degeneracies of the transfer matrix of the eight-vertex model has been discussed [16, 17]. There are also some relevant papers [20, 21].

The purpose of this paper is to formulate a family of operators commuting with the XXZ Hamiltonian with twisted boundary conditions. They may explain the level crossings observed in the spectral flows with respect to the twist angle. Here we generalize the approach given in [12], and give some extended results. As an illustration, we show that when $\Phi = \pi$ and $q$ is a primitive $2N$th root of unity, the operators commuting with the twisted XXZ Hamiltonian generate the $sl_2$ loop algebra for the sector $S^Z = N/2 \,(\mathrm{mod}\ N)$. Furthermore, in the sector $S^Z = N/2 \,(\mathrm{mod}\ N)$, we explicitly show the defining relations of the $sl_2$ loop algebra for the case of the periodic boundary conditions where $L$ is odd and $q$ is a primitive $N$th root of unity with odd $N$. When $N = 3$, it is exactly the case in which various combinatorial formulae were discussed recently [22–25].

The content of the paper consists of the following: in section 2 we review the $sl_2$ loop algebra symmetry of the periodic XXZ spin chain [12]. In section 3 we introduce the transfer matrix of the six-vertex model under twisted boundary conditions. We also discuss some useful techniques such as the gauge transformation and the crossing symmetry. In section 4 we show commutation relations of the twisted transfer matrix with some powers of the quantum group generators, which are fundamental in the paper. We also show those of the inhomogeneous case. In section 5 we discuss operators commuting with the twisted transfer matrix when $q$ is a root of unity. We also discuss the special cases where we can explicitly check the defining relations of $sl_2$ loop algebra. Finally we note a connection to the eight-vertex model.

## 2. The loop algebra symmetry of the periodic XXZ spin chain

Let us review the $sl_2$ loop algebra symmetry of the XXZ spin chain under the periodic boundary conditions and introduce the quantum group $U_q(sl_2)$. The generators $S^{\pm}$ and $S^Z$ satisfy the defining relations

$$[S^+, S^-] = \frac{q^{2S^Z} - q^{-2S^Z}}{q - q^{-1}} \qquad [S^Z, S^{\pm}] = \pm S^{\pm} \tag{4}$$

with the comultiplication $\Delta$ given by

$$\Delta(S^{\pm}) = S^{\pm} \otimes q^{-S^Z} + q^{S^Z} \otimes S^{\pm} \qquad \Delta(S^Z) = S^Z \otimes I + I \otimes S^Z. \tag{5}$$

Here, the parameter $q$ is generic. In fact, we may consider $U_q(L(sl_2))$, i.e., the $q$ analogue of the universal enveloping algebra of the $sl_2$ loop algebra. For simplicity, however, we only consider $U_q(sl_2)$ in the paper.

Let $V$ denote a two-dimensional vector space over $\mathbf{C}$. On the $L$th tensor product space $V^{\otimes L}$, the generators $S^{\pm}$ and $S^Z$ are given by

$$\begin{aligned} q^{S^Z} &= q^{\sigma^Z/2} \otimes \cdots \otimes q^{\sigma^Z/2} \\ S^{\pm} &= \sum_{j=1}^{L} q^{\sigma^Z/2} \otimes \cdots \otimes q^{\sigma^Z/2} \otimes \sigma_j^{\pm} \otimes q^{-\sigma^Z/2} \otimes \cdots \otimes q^{-\sigma^Z/2} = \sum_{j=1}^{L} S_j^{\pm}. \end{aligned} \tag{6}$$

Here $S_j^{\pm}$ denotes the $j$th term in the sum (6). Considering the automorphism of $U_q(L(sl_2))$, we introduce the following operators:

$$T^{\pm} = \sum_{j=1}^{L} q^{-\sigma^Z/2} \otimes \cdots \otimes q^{-\sigma^Z/2} \otimes \sigma_j^{\pm} \otimes q^{\sigma^Z/2} \otimes \cdots \otimes q^{\sigma^Z/2} = \sum_{j=1}^{L} T_j^{\pm}. \tag{7}$$

Let us denote by $S^{\pm(n)}$ and $T^{\pm(n)}$ the following operators:

$$S^{\pm(n)} = (S^{\pm})^n/[n]! \qquad T^{\pm(n)} = (T^{\pm})^n/[n]!. \tag{8}$$

Here $n$ is a positive integer, and $[n]$ and $[n]!$ denote the $q$-integer $[n] = (q^n - q^{-n})/(q - q^{-1})$ and the $q$-factorial $[n]! = [n][n-1]\cdots[1]$, respectively. Then, we have

$$\begin{aligned} S^{\pm(n)} = \sum_{1\leqslant j_1<\cdots<j_n\leqslant L} & q^{\frac{n}{2}\sigma^Z} \otimes \cdots \otimes q^{\frac{n}{2}\sigma^Z} \otimes \sigma_{j_1}^{\pm} \otimes q^{\frac{(n-2)}{2}\sigma^Z} \\ & \otimes \cdots \otimes q^{\frac{(n-2)}{2}\sigma^Z} \otimes \sigma_{j_2}^{\pm} \otimes q^{\frac{(n-4)}{2}\sigma^Z} \otimes \cdots \otimes \sigma_{j_n}^{\pm} \otimes q^{-\frac{n}{2}\sigma^Z} \otimes \cdots \otimes q^{-\frac{n}{2}\sigma^Z} \end{aligned} \tag{9}$$

$$\begin{aligned} T^{\pm(n)} = \sum_{1\leqslant j_1<\cdots<j_n\leqslant L} & q^{-\frac{n}{2}\sigma^Z} \otimes \cdots \otimes q^{-\frac{n}{2}\sigma^Z} \otimes \sigma_{j_1}^{\pm} \otimes q^{-\frac{(n-2)}{2}\sigma^Z} \\ & \otimes \cdots \otimes q^{-\frac{(n-2)}{2}\sigma^Z} \otimes \sigma_{j_2}^{\pm} \otimes q^{-\frac{(n-4)}{2}\sigma^Z} \otimes \cdots \otimes \sigma_{j_n}^{\pm} \otimes q^{\frac{n}{2}\sigma^Z} \otimes \cdots \otimes q^{\frac{n}{2}\sigma^Z}. \end{aligned} \tag{10}$$

Let the symbol $\tau_{6V}(v)$ denote the transfer matrix of the six-vertex model. We now take the parameter $q$ as a root of unity. We consider the limit of sending $q$ to a root of unity: $q^{2N} = 1$. Then we can show the (anti) commutation relations [12] in the sector of $S^Z \equiv 0 \pmod N$

$$S^{\pm(N)}\tau_{6V}(v) = q^N \tau_{6V}(v) S^{\pm(N)} \qquad T^{\pm(N)}\tau_{6V}(v) = q^N \tau_{6V}(v) T^{\pm(N)}. \tag{11}$$

Since the XXZ Hamiltonian $H_{\mathrm{XXZ}}$ is given by the logarithmic derivative of the transfer matrix, we have in the sector $S^Z \equiv 0 \pmod N$

$$[S^{\pm(N)}, H_{\mathrm{XXZ}}] = [T^{\pm(N)}, H_{\mathrm{XXZ}}] = 0. \tag{12}$$

Let us now consider the algebra generated by the operators [12]. When $q$ is a primitive $2N$th root of unity with $N$ even, or a primitive $N$th root of unity with $N$ odd, we consider the following identification [12]:

$$E_0^+ = S^{+(N)} \qquad E_0^- = S^{-(N)} \qquad E_1^+ = T^{-(N)} \qquad E_1^- = T^{+(N)} \qquad H_0 = -H_1 = \frac{2}{N} S^Z. \tag{13}$$

Using the automorphism

$$\theta\left(E_0^{\pm}\right) = E_1^{\pm} \qquad \theta(H_0) = H_1 \tag{14}$$

we may take the following identification:

$$E_0^+ = T^{-(N)} \qquad E_0^- = T^{+(N)} \qquad E_1^+ = S^{+(N)} \qquad E_1^- = S^{-(N)} \qquad -H_0 = H_1 = \frac{2}{N} S^Z. \tag{15}$$

When $q$ is a primitive $2N$th root of unity with $N$ odd, we may put as follows:

$$E_0 = \mathrm{i}T^{-(N)} \qquad E_1 = \mathrm{i}S^{+(N)} \qquad F_0 = \mathrm{i}T^{+(N)} \qquad F_1 = \mathrm{i}S^{-(N)} \qquad -H_0 = H_1 = \frac{2}{N} S^Z. \tag{16}$$

It is shown in [12] that the operators $E_j^{\pm}$, $H_j$ for $j = 0, 1$, satisfy the defining relations of the algebra $U(L(sl_2))$. They are given explicitly by the following:

$$H_0 + H_1 = 0 \qquad [H_i, E_j] = a_{ij} E_j \qquad [H_i, F_j] = -a_{ij} F_j \qquad (i, j = 0, 1) \tag{17}$$

$$[E_i, F_j] = \delta_{ij} H_j \qquad (i, j = 0, 1) \tag{18}$$

$$[E_i, [E_i, [E_i, E_j]]] = 0 \qquad [F_i, [F_i, [F_i, F_j]]] = 0 \qquad (i, j = 0, 1, i \neq j). \tag{19}$$

Here, the Cartan matrix $(a_{ij})$ of $A_1^{(1)}$ is defined by

$$\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \tag{20}$$

Relations (17) hold for generic $q$, while the higher Serre relations (19) hold if $q$ is a primitive $2N$th root of unity or a primitive $N$th root of unity with $N$ odd [12, 26]. Relation (18) holds if $q$ is a primitive $2N$th root of unity with $N$ even or a primitive $N$th root of unity with $N$ odd. When $q$ is a primitive $2N$ th root of unity with $N$ odd, we take the identification (16) with the imaginary factors.

## 3. Twisted transfer matrix of the six-vertex model

### *3.1. Boltzmann weights*

Let us consider the configuration around the vertex as shown in figure 1. Variables $a$, $b$, $c$ and $d$ are defined on the edges at the vertex, and they take value 1 or 2. The value 1 corresponds to a polarization vector that is in the upward or rightward direction, and the value 2 to a polarization vector in the downward or leftward direction. We assign the Boltzmann weight $X_{bd}^{ac}(u)$ to the configuration in figure 1. The weight vanishes unless $a + c = b + d$, which we

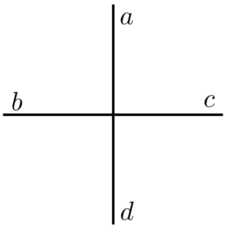


**Figure 1.** Vertex configuration for the Boltzmann weight $X_{bd}^{ac}(u)$. The spectral parameter $u$ corresponds to the angle between the two lines $b$ to $c$ and $d$ to $a$, which is important to the Yang–Baxter equation (25).

call the 'ice rule' or the 'charge conservation'. All the nonzero Boltzmann weights are given by

$$\begin{aligned} &X_{11}^{11}(u) = X_{22}^{22}(u) = \sinh(2\eta + u) \qquad X_{21}^{12}(u) = X_{12}^{21}(u) = \sinh u \\ &X_{12}^{12}(u) = X_{21}^{21}(u) = \sinh 2\eta. \end{aligned} \tag{21}$$

Here $q = \exp(2\eta)$, and $u$ is the spectral parameter.

We define operators $X_j(u)$ for $j = 0, 1, \ldots, L-1$ by

$$X_j(u) = \sum_{a,b,c,d=1,2} X_{bd}^{ac}(u) I_0 \otimes I_1 \otimes \cdots \otimes I_{j-1} \otimes E_j^{ab} \otimes E_{j+1}^{cd} \otimes I_{j+2} \otimes \cdots \otimes I_L \tag{22}$$

where $I$ denotes the identity matrix and $E^{ab}$ denotes the matrix

$$(E^{ab})_{c,d} = \delta_{a,c}\delta_{b,d} \qquad \text{for} \quad c, d = 1, 2. \tag{23}$$

It is easy to show that the operators $X_j(u)$ constructed from the Boltzmann weights (21) satisfy the Yang–Baxter equation in the following:

$$X_j(u)X_{j+1}(u+v)X_j(v) = X_{j+1}(v)X_j(u+v)X_{j+1}(u). \tag{24}$$

In terms of the Boltzmann weights the operator relation (24) is written as follows:

$$\sum_{\alpha,\beta,\gamma} X_{\alpha\gamma}^{a_1a_2}(u) X_{\beta b_3}^{\gamma a_3}(u+v) X_{b_1b_2}^{\alpha\beta}(v) = \sum_{\alpha,\beta,\gamma} X_{\beta\alpha}^{a_2a_3}(v) X_{b_1\gamma}^{a_1\beta}(u+v) X_{b_2b_3}^{\gamma\alpha}(u). \tag{25}$$

### *3.2. The six-vertex transfer matrix with the twisted boundary conditions*

Recall that the operators $X_j(u)$ are acting on the tensor product $V^{\otimes(L+1)} = V_0 \otimes V_1 \otimes \cdots \otimes V_L$. We now define the twisted transfer matrix of the six-vertex model by

$$\tau(u;\phi) = \mathrm{tr}_0\big(q^{\phi\sigma_0^Z} X_{L-1}(u) \cdots X_1(u) X_0(u)\big) \tag{26}$$

where the symbol $\mathrm{tr}_0$ denotes the trace over the 0th space $V_0$.

The logarithmic derivative of the twisted transfer matrix leads to the XXZ Hamiltonian $\mathcal{H}_{\mathrm{XXZ}}(\phi)$ under the twisted boundary conditions

$$\begin{aligned} \sinh 2\eta \times \frac{\mathrm{d}}{\mathrm{d}u} \log \tau(u;\phi)|_{u=0} &= \sum_{j=1}^{L-1} \big(2\sigma_j^+\sigma_{j+1}^- + 2\sigma_j^-\sigma_{j+1}^+ + \cosh 2\eta \sigma_j^Z \sigma_{j+1}^Z\big) \\ &\quad + q^{-2\phi} 2\sigma_L^+\sigma_1^- + q^{2\phi} 2\sigma_L^-\sigma_1^+ + \cosh 2\eta \sigma_L^Z\sigma_1^Z + L\cosh 2\eta \\ &= \mathcal{H}_{\mathrm{XXZ}}(\phi)/J + L\Delta \end{aligned} \tag{27}$$

where $\Delta = \cosh 2\eta$ and the twist angle $\Phi$ is related to $\phi$ through the relation

$$q^{2\phi} = \exp(\mathrm{i}\Phi). \tag{28}$$

We should note that the twisted transfer matrix (26) is different from that of [12]: the logarithmic derivative of the transfer matrix $T^{\mathrm{DFM}}(v;\phi)$ of [12] leads to the XXZ Hamiltonian under the periodic boundary conditions for any $\phi$, while that of (26) gives the twisted XXZ Hamiltonian (27).

Let the symbol $\Pi^{(12\cdots L)}$ denote the shift operator defined by

$$\Pi^{(12\cdots L)} e_1 \otimes \cdots \otimes e_L = e_L \otimes e_1 \otimes \cdots \otimes e_{L-1}. \tag{29}$$

Then, the twisted transfer matrix $\tau(u;\phi)$ gives the twisted shift operator $\tau(0;\phi) = \sinh^L 2\eta \Pi(\phi)$, where $\Pi(\phi)$ is defined by

$$\Pi(\phi) = q^{\phi\sigma_1^Z}\Pi^{(12\cdots L)} = \Pi^{(12\cdots L)} q^{\phi\sigma_L^Z}. \tag{30}$$

### *3.3. Gauge transformations and the crossing symmetry*

We introduce the following transformation on the Boltzmann weights of the six-vertex model [27]:

$$X_{bd}^{ac}(u) \to \tilde{X}_{bd}^{ac}(u) = \epsilon^{a+b}\exp\left(\kappa(a+b-c-d)u/2\right) X_{bd}^{ac}(u). \tag{31}$$

Here $\kappa$ is arbitrary and $\epsilon = \pm 1$. We can show that the transformed weights $\tilde{X}_{bd}^{ac}(u)$ satisfy the Yang–Baxter equation (25) if $X_{bd}^{ac}(u)$ satisfy it. We call (31) a gauge transformation. Let $\tilde{X}_j(u)$ denote the operator defined by (22) with the $X_{bd}^{ac}(u)$ replaced with $\tilde{X}_{bd}^{ac}(u)$. Then, $\tilde{X}_j(u)$ also satisfy the Yang–Baxter equation (24).

Let us discuss the gauge transformation (31) with $\epsilon = 1$. It has two important properties. First, the transfer matrix $\tau(u;\phi)$ is invariant under the gauge transformation due to the charge conservation. Second, when $\kappa = \pm 1$, $\tilde{X}_j(u)$ can be expressed in terms of the generator of the Temperley–Lieb algebra [27]. Let $X_{bd}^{\pm ac}(u)$ denote the transformed weight $\tilde{X}_{bd}^{ac}(u)$ with $\kappa = \pm 1$, respectively. In terms of the Boltzmann weights, we have the following decomposition [27]:

$$X_{bd}^{\pm ac}(u) = \sinh(u+2\eta)\delta_{a,b}\delta_{c,d} + \sinh u\, r_a^{\pm} r_b^{\pm}\delta_{a,\bar{c}}\delta_{b,\bar{d}}. \tag{32}$$

Here $\bar{a}$ denotes the conjugate of $a$, which is defined by $\bar{1} = 2$ and $\bar{2} = 1$. The quantities $r_j^{\pm}$ ($j = 1, 2$) are defined by [28]

$$r_1^{\pm} = \mathrm{i}\exp(\mp\eta) \qquad r_2^{\pm} = -\mathrm{i}\exp(\pm\eta). \tag{33}$$

The weights $X_{b,d}^{\pm a,c}(u)$ have the following symmetry [28]:

$$X_{b,d}^{\pm a,c}(u) = -r_b^{\pm} r_{\bar{c}}^{\pm} X_{d,\bar{c}}^{\pm\bar{b},a}(-2\eta - u) = -r_a^{\pm} r_{\bar{d}}^{\pm} X_{\bar{a},b}^{\pm c,\bar{d}}(-2\eta - u). \tag{34}$$

We call it the crossing symmetry.

### *3.4. The Temperley–Lieb decomposition of $X_j(u)$*

Let us introduce the following operators:

$$U_j^{\pm} = \sum_{a,b,c,d=1,2} r_a^{\pm} r_b^{\pm}\delta_{a,\bar{c}}\delta_{b,\bar{d}} I_0 \otimes I_1 \otimes \cdots \otimes I_{j-1} \otimes E_j^{ab} \otimes E_{j+1}^{cd} \otimes I_{j+2} \otimes \cdots \otimes I_L. \tag{35}$$

The $U_j^{\pm}$ satisfy the defining relations of the Temperley–Lieb algebra [29, 30]:

$$\left(U_j^{\pm}\right)^2 = Q^{1/2} U_j^{\pm} \qquad U_j^{\pm} U_{j+1}^{\pm} U_j^{\pm} = U_j^{\pm} \qquad U_{j+1}^{\pm} U_j^{\pm} U_{j+1}^{\pm} = U_{j+1}^{\pm} \tag{36}$$

for $j = 1, 2, \ldots, L-1$. Here $Q^{1/2}$ is given by $Q^{1/2} = -(q + q^{-1})$. We call $U_j^{\pm}$ the Temperley–Lieb operators.

In terms of the Temperley–Lieb operators, the operator $X_j^{\pm}(u)$ can be expressed as follows [30]:

$$X_j^{\pm}(u) = \sinh(u + 2\eta)I + \sinh u U_j^{\pm}. \tag{37}$$

The decomposition (37) corresponds to (32) for the Boltzmann weights.

The Temperley–Lieb operators $U_j^{+}$ commute with the generators of $U_q(sl_2)$. We can show $\left[S^{\pm}, U_j^{+}\right] = 0$ for $j = 0, 1, \ldots, L-1$. It can be important, since the XXZ Hamiltonian can be expressed in terms of $U_j^{+}$ and a boundary term [31].

It is known that the periodic XXZ spin chain does not commute with $U_q(sl_2)$. In fact, the generators $S^{\pm}$ do not commute with the periodic XXZ Hamiltonian, since they are not compatible with the periodic boundary conditions. When $q$ is a root of unity, however, some powers of $S^{\pm}$ can commute with it [31]. As a matter of fact, the observation can be developed much further [12]. We first note that $U_j^{+}$ commute with $S^{\pm}$ while they do not with $T^{\pm}$, and $U_j^{-}$ commute with $T^{\pm}$ while they do not with $S^{\pm}$. When $q^{2N} = 1$, we can show that the transfer matrix commutes with $S^{\pm(N)}$ and $T^{\pm(N)}$ simultaneously, and they generate the $sl_2$ loop algebra [12].

## 4. Relations of the transfer matrix for *q* generic

### *4.1. Decomposition of the transfer matrix*

Let us consider the gauge transformations (31) with $\kappa = \pm 1$ and $\epsilon = 1$. Due to the charge conservation, they do not change any of the matrix elements of the transfer matrix (26). Thus, we have

$$\tau(u;\phi) = \mathrm{tr}_0\big(q^{\phi\sigma_0^Z} X_{L-1}^{\pm}(u) \cdots X_1^{\pm}(u) X_0^{\pm}(u)\big). \tag{38}$$

We denote by $\tau^{\pm}(u;\phi)$ the right-hand side of (38). Putting the decomposition (37) for $X_0^{\pm}(u)$ into $\tau^{\pm}(u;\phi)$, and making use of the crossing symmetry (34), we can show the following:

$$\tau(u;\phi) = \Pi(\phi) X_{LL}^{\pm}(u) + X_{RR}^{\pm}(u)\Pi(\phi)^{-1} \tag{39}$$

where the symbols $X_{LL}^{\pm}(u)$ and $X_{RR}^{\pm}(u)$ are given by

$$\begin{aligned} X_{LL}^{\pm}(u) &= \sinh(u + 2\eta) X_{L-1}^{\pm}(u) \cdots X_2^{\pm}(u) X_1^{\pm}(u) \\ X_{RR}^{\pm}(u) &= (-1)^L \sinh(u) X_1^{\pm}(-2\eta - u) X_2^{\pm}(-2\eta - u) \cdots X_{L-1}^{\pm}(-2\eta - u). \end{aligned} \tag{40}$$

We remark that $X_j^{+}(u)$ commute with $S^{\pm}$ while $X_j^{-}(u)$ commute with $T^{\pm}$. When $\phi = 0$, (39) is reduced to that of the periodic one [12].

### *4.2. Transformations of the twisted shift operator*

We can show the following relations of $(S^{\pm})^n$ and $(T^{\pm})^n$ for generic $q$:

$$\begin{aligned} \Pi(\phi)(S^{\pm})^n \Pi(\phi)^{-1} &= q^{-n\sigma_1^Z}\left\{(S^{\pm})^n + q^{\mp(n-1)}[n](S^{\pm})^{n-1} S_1^{\pm}(q^{2(S^Z \pm n \pm \phi)} - 1)\right\} \\ \Pi(\phi)^{-1}(S^{\pm})^n \Pi(\phi) &= q^{n\sigma_L^Z}\left\{(S^{\pm})^n + q^{\pm(n-1)}[n](S^{\pm})^{n-1} S_L^{\pm}(q^{-2(S^Z \pm n \pm \phi)} - 1)\right\} \\ \Pi(\phi)(T^{\pm})^n \Pi(\phi)^{-1} &= q^{n\sigma_1^Z}\left\{(T^{\pm})^n + q^{\pm(n-1)}[n](T^{\pm})^{n-1} T_1^{\pm}(q^{-2(S^Z \pm n \mp \phi)} - 1)\right\} \\ \Pi(\phi)^{-1}(T^{\pm})^n \Pi(\phi) &= q^{-n\sigma_L^Z}\left\{(T^{\pm})^n + q^{\mp(n-1)}[n](T^{\pm})^{n-1} T_L^{\pm}(q^{2(S^Z \pm n \mp \phi)} - 1)\right\}. \end{aligned} \tag{41}$$

Let us briefly discuss the derivation. We first consider the following:

$$\begin{aligned}\Pi(\phi)(S^{\pm})^n\Pi(\phi)^{-1} &= (\Pi(\phi)S^{\pm}\Pi(\phi)^{-1})^n \\ &= \left\{\left(S^{\pm}-S_1^{\pm}\right)q^{-\sigma_1^Z}+q^{2\phi}S_1^{\pm}q^{2S^Z-\sigma_1^Z}\right\}^n.\end{aligned} \tag{42}$$

Here we denote by $A$ and $q^{\pm 2\phi}B$ the first and second terms of the right-hand side, respectively. Then we can show the following:

$$(A+q^{\pm 2\phi}B)^n = A^n + q^{\pm 2\phi}\sum_{j=0}^{n-1}A^{n-1-j}BA^j. \tag{43}$$

Thus, we can derive expression (41) through the following:

$$\begin{aligned}A^n &= q^{-n\sigma_1^Z}\left(S^{\pm}-S_1^{\pm}\right)^n = q^{-n\sigma_1^Z}\left((S^{\pm})^n - q^{\mp(n-1)}[n](S^{\pm})^{n-1}S_1^{\pm}\right)\\ A^{n-1-j}BA^j &= q^{\pm 2(j+1)}q^{-n\sigma_1^Z}(S^{\pm})^{n-1}S_1^{\pm}q^{2S^Z}.\end{aligned} \tag{44}$$

### *4.3. Relations of the twisted transfer matrix*

Using relations (41), we can derive the following relations of the twisted transfer matrix for generic $q$:

$$\begin{aligned}(S^{\pm})^n\tau(u;\phi) &= \tau(u;\phi+n)(S^{\pm})^n + q^{\pm(n-1)}[n]\Pi(\phi+n)(S^{\pm})^{n-1}S_L^{\pm}(q^{-2(S^Z\pm n\pm\phi)}-1)X_{LL}^{+}(u)\\ &\quad + q^{\mp(n-1)}[n]X_{RR}^{+}(u)\Pi(\phi+n)^{-1}(S^{\pm})^{n-1}S_1^{\pm}(q^{2(S^Z\pm n\pm\phi)}-1)\\ (T^{\pm})^n\tau(u;\phi) &= \tau(u;\phi-n)(T^{\pm})^n + q^{\mp(n-1)}[n]\Pi(\phi-n)(T^{\pm})^{n-1}T_L^{\pm}(q^{2(S^Z\pm n\mp\phi)}-1)X_{LL}^{-}(u)\\ &\quad + q^{\pm(n-1)}[n]X_{RR}^{-}(u)\Pi(\phi-n)^{-1}(T^{\pm})^{n-1}T_1^{\pm}(q^{-2(S^Z\pm n\mp\phi)}-1).\end{aligned} \tag{45}$$

Here we note that $S^{\pm}$ commute with $X_{LL}^{+}(u)$ and $X_{RR}^{+}(u)$, and also that $T^{\pm}$ commute with $X_{LL}^{-}(u)$ and $X_{RR}^{-}(u)$. Some different forms of the commutation relations are given in appendix A.

We remark that some relations of $S^{\pm(N)}$ with the transfer matrix have been investigated [32].

### *4.4. Inhomogeneous case of the twisted transfer matrix*

We define the inhomogeneous transfer matrix of the six-vertex model under the twisted boundary conditions by

$$\tau(u;\phi;\{w_j\}) = \mathrm{tr}_0\left(q^{\phi\sigma_0^Z}X_{L-1}(u-w_{L-1})\right)\cdots X_1(u-w_1)X_0(u-w_0)). \tag{46}$$

Here $w_j$ are called inhomogeneous parameters. The twisted transfer matrix (46) is called inhomogeneous. Under the gauge transformation (31) with $\epsilon = 1$, $\tau(u;\phi;\{w_j\})$ is mapped to $\tilde{\tau}(u;\phi;\{w_j\})$ as follows:

$$\tau(u;\phi;\{w_j\}) = V(\kappa)\tilde{\tau}(u;\phi;\{w_j\})V(\kappa)^{-1} \tag{47}$$

where $V(\kappa)$ is given by the diagonal matrix

$$(V(\kappa))_{b_1,b_2,\ldots,b_L}^{a_1,a_2,\ldots,a_L} = \exp\left(\kappa\sum_{j=1}^{L}w_ja_j\right)\delta_{a_1,b_1}\delta_{a_2,b_2}\cdots\delta_{a_L,b_L}. \tag{48}$$

Let $\tau^{\pm}(u;\phi;\{w_j\})$ denote the transformed transfer matrices $\tilde{\tau}(u;\phi;\{w_j\})$ with $\kappa = \pm 1$, respectively. We write $V(\pm 1)$ by $V^{\pm}$. We thus have

$$\tau(u;\phi;\{w_j\}) = V^{\pm}\tau^{\pm}(u;\phi;\{w_j\})V^{\mp}. \tag{49}$$

We therefore define $\tilde{S}^{\pm}$ and $\tilde{T}^{\pm}$ by

$$\tilde{S}^{\pm} = V^{+}S^{\pm}V^{-} \qquad \tilde{T}^{\pm} = V^{-}T^{\pm}V^{+}. \tag{50}$$

The transfer matrices $\tau^{\pm}(u;\phi;\{w_j\})$ can be decomposed as

$$\tau^{\pm}(u;\phi;\{w_j\}) = \Pi(\phi)X^{\pm}_{LL}(u;\{w_j\}) + X^{\pm}_{RR}(u;\{w_j\})\Pi(\phi)^{-1} \tag{51}$$

where $X^{\pm}_{LL}(u;\{w_j\})$ and $X^{\pm}_{RR}(u;\{w_j\})$ are given by

$$\begin{aligned} X^{\pm}_{LL}(u;\{w_j\}) &= \sinh(u+2\eta-w_0)X^{\pm}_{L-1}(u-w_{L-1})\cdots X^{\pm}_{1}(u-w_1)\\ X^{\pm}_{RR}(u;\{w_j\}) &= (-1)^L\sinh(u-w_0)X^{\pm}_{1}(-2\eta-u+w_1)\\ &\quad\times X^{\pm}_{2}(-2\eta-u+w_2)\cdots X^{\pm}_{L-1}(-2\eta-u+w_{L-1}). \end{aligned} \tag{52}$$

In the same way as (45) we can show

$$\begin{aligned} (S^{\pm})^n\tau^{+}(u;\phi;\{w_j\}) &= \tau^{+}(u;\phi+n;\{w_j\})(S^{\pm})^n\\ &\quad + q^{\pm(n-1)}[n]\Pi(\phi+n)(S^{\pm})^{n-1}S^{\pm}_L(q^{-2(S^Z\pm n\pm\phi)}-1)X^{+}_{LL}(u;\{w_j\})\\ &\quad + q^{\mp(n-1)}[n]X^{+}_{RR}(u;\{w_j\})\Pi(\phi+n)^{-1}(S^{\pm})^{n-1}S^{\pm}_1(q^{2(S^Z\pm n\pm\phi)}-1)\\ (T^{\pm})^n\tau^{-}(u;\phi;\{w_j\}) &= \tau^{-}(u;\phi-n;\{w_j\})(T^{\pm})^n\\ &\quad + q^{\mp(n-1)}[n]\Pi(\phi-n)(T^{\pm})^{n-1}T^{\pm}_L(q^{2(S^Z\pm n\mp\phi)}-1)X^{-}_{LL}(u;\{w_j\})\\ &\quad + q^{\pm(n-1)}[n]X^{-}_{RR}(u;\{w_j\})\Pi(\phi-n)^{-1}(T^{\pm})^{n-1}T^{\pm}_1(q^{-2(S^Z\pm n\mp\phi)}-1). \end{aligned} \tag{53}$$

Through relation (49) we can derive commutation or anti-commutation relations for the inhomogeneous twisted transfer matrix $\tau(u;\phi;\{w_j\})$. Let $|k\rangle\rangle$ denote such a vector with $S^Z = k$. Then we have

$$\begin{aligned} (\tilde{S}^{\pm})^n\tau(u;\phi;\{w_j\})|k\rangle\rangle &= \tau(u;\phi+n;\{w_j\})(\tilde{S}^{\pm})^n|k\rangle\rangle \qquad \text{when} \quad q^{2(k\pm n\pm\phi)} = 1\\ (\tilde{T}^{\pm})^n\tau(u;\phi;\{w_j\})|k\rangle\rangle &= \tau(u;\phi-n;\{w_j\})(\tilde{T}^{\pm})^n|k\rangle\rangle \qquad \text{when} \quad q^{2(k\pm n\mp\phi)} = 1. \end{aligned} \tag{54}$$

We note that for the case of $\phi = 0$, the inhomogeneous result was addressed at the end of [12].

## 5. The loop algebra symmetry

### *5.1. Operators commuting with the twisted transfer matrix*

Let us assume that $q$ is a root of unity $q^{2N} = 1$. We denote by $|k\rangle$ such a vector that has a fixed $S^Z$ value and it is equivalent to $k$ mod $N$: $S^Z \equiv k$ (mod $N$). Here $k$ is an integer or a half-integer. Let $m$ and $n$ be two non-negative integers such that $m \equiv n$ (mod $N$). Then we have

$$\begin{aligned} (S^{\pm})^m(T^{\mp})^n\tau(u;\phi)|\pm n\mp\phi\rangle &= \tau(u;\phi+m-n)(S^{\pm})^m(T^{\mp})^n|\pm n\mp\phi\rangle\\ (T^{\pm})^m(S^{\mp})^n\tau(u;\phi)|\pm n\pm\phi\rangle &= \tau(u;\phi-m+n)(T^{\pm})^m(S^{\mp})^n|\pm n\pm\phi\rangle. \end{aligned} \tag{55}$$

Relations (55) are derived from (45). We note that when $m = n$ expressions (55) are also valid for the case of $q$ generic.

Let us denote by $\theta(n)$ the least non-negative integer which is equivalent to $n$ mod $N$. In the sector $S^Z \equiv \ell$ (mod $N$) we have commutation relations $X\tau(u,p) = \tau(u,p)X$, where $X$ are given by

$$(S^{\pm})^{\theta(\pm\ell+p)}(T^{\mp})^{\theta(\pm\ell+p)} \qquad (T^{\pm})^{\theta(\pm\ell-p)}(S^{\mp})^{\theta(\pm\ell-p)} \tag{56}$$

and commutation or anti-commutation relations $X\tau(u,\,p) = q^N\tau(u,\,p)X$, where $X$ are given by

$$
\begin{array}{ll}
(S^{\pm})^{\theta(\pm\ell+p)+N}(T^{\mp})^{\theta(\pm\ell+p)} & (S^{\pm})^{\theta(\pm\ell+p)}(T^{\mp})^{\theta(\pm\ell+p)+N} \\
(T^{\pm})^{\theta(\pm\ell-p)+N}(S^{\mp})^{\theta(\pm\ell-p)} & (T^{\pm})^{\theta(\pm\ell-p)}(S^{\mp})^{\theta(\pm\ell-p)+N}.
\end{array}
\tag{57}
$$

Here we assume that $\ell$ and $p$ are such integers or half-integers that $\ell \pm p$ are integers.

In the sector $S^Z \equiv \ell \pmod N$, the operators (56) and (57) commute with the twisted XXZ Hamiltonian $\mathcal{H}_{\mathrm{XXZ}}(\phi)$ with $\phi \equiv p \pmod N$. We recall that $\mathcal{H}_{\mathrm{XXZ}}(\phi)$ is given by the logarithmic derivative of the twisted transfer matrix $\tau(u;\phi)$. For instance, we can show that $S^{\pm(N)}$ and $T^{\pm(N)}$ commute with the anti-periodic XXZ Hamiltonian in the sector $S^Z \equiv N/2 \pmod N$, when $q$ is a primitive $2N$th root of unity. Putting $\ell = N/2$ and $p = N/2$ in (57), we have

$$
[S^{\pm(N)}, H_{\mathrm{XXZ}}(N/2)]\,|\,N/2\rangle = [T^{\pm(N)}, H_{\mathrm{XXZ}}(N/2)]\,|\,N/2\rangle = 0. \tag{58}
$$

### 5.2. *Symmetry operators for the inhomogeneous twisted transfer matrix*

From equations (54), in the sector $S^Z \equiv \ell \pmod N$, we have the commutation relations $X\tau(u;\,p;\,\{w_j\}) = \tau(u;\,p;\,\{w_j\})X$ where $X$ are given by

$$
(\tilde{S}^{\pm})^{\theta(\pm\ell+p)}(\tilde{T}^{\mp})^{\theta(\pm\ell+p)} \qquad (\tilde{T}^{\pm})^{\theta(\pm\ell-p)}(\tilde{S}^{\mp})^{\theta(\pm\ell-p)} \tag{59}
$$

and commutation or anti-commutation relations $X\tau(u;\,p;\,\{w_j\}) = q^N\tau(u;\,p;\,\{w_j\})X$ where $X$ are given by

$$
\begin{array}{ll}
(\tilde{S}^{\pm})^{\theta(\pm\ell+p)+N}(\tilde{T}^{\mp})^{\theta(\pm\ell+p)} & (\tilde{S}^{\pm})^{\theta(\pm\ell+p)}(\tilde{T}^{\mp})^{\theta(\pm\ell+p)+N} \\
(\tilde{T}^{\pm})^{\theta(\pm\ell-p)+N}(\tilde{S}^{\mp})^{\theta(\pm\ell-p)} & (\tilde{T}^{\pm})^{\theta(\pm\ell-p)}(\tilde{S}^{\mp})^{\theta(\pm\ell-p)+N}.
\end{array}
\tag{60}
$$

Here we recall that $\ell$ and $p$ are such integers or half-integers that $\ell \pm p$ are integers.

### 5.3. *The $sl_2$ loop algebra at $\Phi = \pi$*

Let us discuss the anti-periodic boundary conditions or the twisted boundary conditions with $\Phi = \pi$. In the sector $S^Z \equiv N/2 \pmod N$, we can show explicitly the defining relations of the $sl_2$ loop algebra. We consider two cases: (i) $q$ is a primitive $2N$th root of unity with $N$ even ($L$ is even); (ii) $q$ is a primitive $2N$th root of unity with $N$ odd ($L$ is odd).

Let us consider the formula for generic $q$

$$
[S^{+(N)}, S^{-(N)}] = \sum_{j=1}^{N} \frac{S^{-(N-j)}S^{+(N-j)}}{[j]!} \prod_{k=0}^{j-1} \frac{q^{2S^Z-k} - q^{-2S^Z+k}}{q - q^{-1}}. \tag{61}
$$

Taking the limit $q^{2N} \to 1$ in the sector $S^Z \equiv N/2 \pmod N$, we have the following for the cases (i) and (ii):

$$
[S^{+(N)}, S^{-(N)}] = [T^{+(N)}, T^{-(N)}] = \frac{2}{N}S^Z. \tag{62}
$$

We thus take the following identification for the cases (i) and (ii):

$$
E_0^+ = T^{-(N)} \qquad E_0^- = T^{+(N)} \qquad E_1^+ = S^{+(N)} \qquad E_1^- = S^{-(N)} \qquad -H_0 = H_1 = \frac{2}{N}S^Z. \tag{63}
$$

Then, we can show that they satisfy the defining relations of the $sl_2$ loop algebra: (17), (18) and (19).

### *5.4. The $sl_2$ loop algebra symmetry of the periodic XXZ spin chain with L odd*

As an application of the (anti-)commutation relations (45), we discuss the case of the periodic boundary conditions with $L$ odd, where $q$ is a primitive $N$th root of unity with $N$ odd. Taking the same identification (63) of the generators, we can show explicitly that they satisfy the defining relations of the $sl_2$ loop algebra in the sector $S^Z \equiv N/2 \pmod N$.

For sectors other than $S^Z \equiv N/2 \pmod N$, we have not explicitly shown the defining relations of the $sl_2$ loop algebra. However, we have a conjecture that some of the operators given in (57) should generate the $sl_2$ loop algebra. By a different method, we can show that the spectral degeneracies related to the $sl_2$ loop algebra also exist in sectors other than $S^Z \equiv N/2 \pmod N$. We can derive it from the general result on the spectral degeneracy of the eight-vertex model [16] and through the XXZ limit of the XYZ spin chain. Some details should be discussed elsewhere.

## Acknowledgments

The author would like to thank A A Belavin, C Korff, K Kudo, B M McCoy and A Nishino for their helpful comments. He would also like to thank the organizers: D Arnaudon, J Avan, L Frappat, E Ragoucy and P Sorba for their kind invitation to the international workshop on Recent Advances in the Theory of Quantum Integrable Systems (RAQIS03), 25–28 March 2003, LAPTH, Annecy-le-Vieux, France. This work is partially supported by a grant-in-aid (no 14702012).

## Appendix. Commutation or anti-commuation relations

We can show the following relations for $q$ generic:

$$\begin{aligned}
\Pi(\phi)(S^{\pm})^n\Pi(\phi)^{-1} &= \left\{(S^{\pm})^n + q^{\mp(n-1)}[n](S^{\pm})^{n-1}S_1^{\pm}(q^{2(S^Z\pm\phi)}-1)\right\}q^{-n\sigma_1^Z} \\
\Pi(\phi)^{-1}(S^{\pm})^n\Pi(\phi) &= \left\{(S^{\pm})^n + q^{\pm(n-1)}[n](S^{\pm})^{n-1}S_L^{\pm}(q^{-2(S^Z\pm\phi)}-1)\right\}q^{n\sigma_L^Z} \\
\Pi(\phi)(T^{\pm})^n\Pi(\phi)^{-1} &= \left\{(T^{\pm})^n + q^{\pm(n-1)}[n](T^{\pm})^{n-1}T_1^{\pm}(q^{-2(S^Z\mp\phi)}-1)\right\}q^{n\sigma_1^Z} \\
\Pi(\phi)^{-1}(T^{\pm})^n\Pi(\phi) &= \left\{(T^{\pm})^n + q^{\mp(n-1)}[n](T^{\pm})^{n-1}T_L^{\pm}(q^{2(S^Z\mp\phi)}-1)\right\}q^{-n\sigma_L^Z}.
\end{aligned} \tag{64}$$

By using (64) we have

$$\begin{aligned}
(S^{\pm})^n\tau(u;\phi) &= \tau(u;\phi+n)(S^{\pm})^n - q^{\mp(n-1)}[n](S^{\pm})^{n-1}S_1^{\pm}(q^{2(S^Z\pm n\pm\phi)}-1)\Pi(\phi)X_{LL}^{+}(u) \\
&\quad - q^{\pm(n-1)}[n]X_{RR}^{+}(u)(S^{\pm})^{n-1}S_L^{\pm}(q^{-2(S^Z\pm n\pm\phi)}-1)\Pi(\phi)^{-1} \\
(T^{\pm})^n\tau(u;\phi) &= \tau(u;\phi-n)(T^{\pm})^n - q^{\pm(n-1)}[n](T^{\pm})^{n-1}T_1^{\pm}(q^{-2(S^Z\pm n\mp\phi)}-1)\Pi(\phi)X_{LL}^{-}(u) \\
&\quad - q^{\mp(n-1)}[n]X_{RR}^{-}(u)(T^{\pm})^{n-1}T_1^{\pm}(q^{2(S^Z\pm n\mp\phi)}-1)\Pi(\phi)^{-1}.
\end{aligned} \tag{65}$$

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